CMS Technical Summary Report #90-4

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July 1989

(Received July 18, 1989)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Air Force Office of
Scientific Research
Washington, DC 20332

National Science
Foundation
Washington, DC 20550

STABILITY OF DISCONTINUOUS SHEARING MOTIONS OF A NON-NEWTONIAN FLUID *

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1. Introduction

~~In~~ this paper, ~~we discuss~~ recent results on the nonlinear stability of discontinuous steady states of a model initial-boundary value problem in one space dimension for incompressible, isothermal shear flow of a non-Newtonian fluid between parallel plates located at $x = \pm 1$, and driven by a constant pressure gradient. The non-Newtonian contribution to the shear stress is assumed to satisfy a simple differential constitutive law. The key feature is a non-monotone relation between the total steady shear stress and steady shear strain rate that results in steady states having, in general, discontinuities in the strain rate. We explain why every solution tends to a steady state as $t \to \infty$, and we identify steady states that are stable; more details and proofs will be presented in [8].

We study the system

$$v_t = S_x \,, \qquad S := T + f\,x, \qquad T := \sigma + v_x \,, \tag{1.1}$$

$$\sigma_t + \sigma = g(v_x) \,, \tag{1.2}$$

on $[0,1] \times [0,\infty)$, with f a fixed positive constant. We impose the boundary conditions

$$S(0,t) = 0 \quad \text{and} \quad v(1,t) = 0, \qquad t \geq 0 \,, \tag{1.3}$$

and the initial conditions

$$v(x,0) = v_0(x), \quad \sigma(x,0) = \sigma_0(x), \qquad 0 \leq x \leq 1 \,; \tag{1.4}$$

accordingly, $S(x,0) = S_0(x) := \sigma_0(x) + v_{0x}(x) + f x$. The function $g : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth, odd, and $\xi\, g(\xi) > 0$, $\xi \neq 0$. In the context of shear flow, v, the velocity

* Supported by the U. S. Army Research Office under Grant DAAL03-87-K-0036 and DAAL03-88-K-0185, the Air Force Office of Scientific Research under Grant AFOSR-87-0191; the National Science Foundation under Grants DMS-8712058, DMS-8620303, DMS-8716132, and a NSF Post Doctoral Fellowship (Pego).

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of the fluid in the channel, and T, the shear stress, are connected through the balance of linear momentum (1.1). The shear stress T is decomposed into a non-Newtonian contribution σ, evolving in accordance with the simple differential constitutive law (1.2), and a viscous contribution v_x. The coefficients of density and Newtonian viscosity are taken as 1, without loss of generality. The flow is assumed to be symmetric about the centerline of the channel. Symmetry dictates the following compatibility restrictions on the initial data:

$$v_0(1) = 0, \qquad S_0(0) = 0, \quad \textit{and} \quad \sigma_0(0) = 0\,; \tag{1.5}$$

they imply that $\sigma(0,t) = v_x(0,t) = 0$, and symmetry is preserved for all time.

The system (1.1)–(1.4) admits steady state solutions $(\overline{v}(x), \overline{\sigma}(x))$ satisfying

$$\bar{S} := g(\bar{v}_x) + \bar{v}_x + fx = 0, \qquad \bar{\sigma} = g(\bar{v}_x) \tag{1.6}$$

on the interval $[0,1]$. In case the function $w(\xi) := g(\xi) + \xi$ is not monotone, there may be multiple values of $\overline{v}_x(x)$ that satisfy (1.6) for some $x's$, thus leading to steady velocity profiles with jumps in the steady velocity gradient $\overline{v}_x$. Our objective is to study the stability of such steady velocity profiles; we also study well-posedness and the convergence of solutions of (1.1)–(1.4) to steady states as $t \to \infty$.

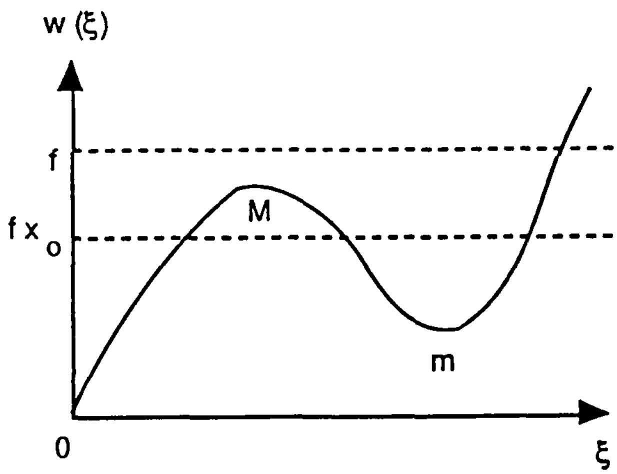

Fig. 1: w vs. ξ.

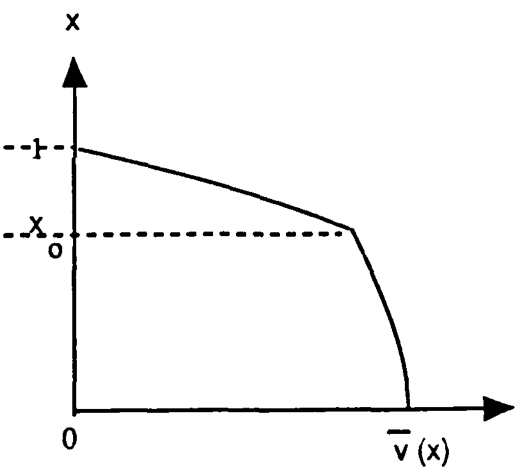

Fig. 2: Velocity profile with a kink; $w(-\overline{v}_x(x)) = fx$.

For simplicity, the function $w(\xi)$ is assumed to have a single loop. The graph of a representative $w(\xi)$ is shown in Fig. 1; in the figure m and M stand for the levels of the bottom and top of the loop, respectively. Our results and techniques can be easily generalized to cover the case when $w(\xi)$ has a finite number of loops. Steady state velocity profiles are constructed as follows: First solve $w(\overline{u}(x)) = fx$ for each $x \in [0,1]$, where $\overline{u} = -\overline{v}_x$. This equation admits a unique solution for $0 \leq fx < m$ or $fx > M$, and three solutions for $m < fx < M$; let $\overline{u}(x)$, $0 \leq x \leq 1$, be a solution. Setting

$$\overline{v}(x) = \int_x^1 \overline{u}(y)dy, \qquad \overline{\sigma}(x) = g(-\overline{u}(x))\,, \tag{1.7}$$

then $(\overline{v}(x), \overline{\sigma}(x))$ satisfy (1.6) and (1.3) for a.e. $x \in [0,1]$ and give rise to a steady state. Clearly, if $f < m$ there is a unique smooth steady state; if $m < f < M$, there is a

unique smooth velocity profile and a multitude of profiles with kinks; finally, if $f > M$, all steady state velocity profiles have kinks. An example of a velocity profile with kinks is shown in Fig. 2.

Problem (1.1)–(1.4) captures certain key features of a class of viscoelastic models that have been proposed to explain the occurence of "spurt" phenomena in non-Newtonian flows. Specifically, for a particular choice of the function g in (1.2), the system under study has the same steady states as the more realistic systems studied in [6] and [7]; the latter, derived from a three-dimensional setting that is restricted to one-dimensional shearing motions, produce non-monotone steady shear stress vs. strain-rate relations of the type shown in Fig. 2. The phenomenon of spurt was apparently first observed by Vinogradov *et al.* [13] in the flow of highly elastic and very viscous non-Newtonian fluids through capillaries or slit-dies. It is associated with a sudden increase in the volumetric flow rate occuring at a critical stress that appears to be independent of the molecular weight. It has been proposed by Hunter and Slemrod [5], using techniques of conservation laws, and more recently by Malkus, Nohel, and Plohr [6] and [7], using numerical simulation and extensive analysis of suitable approximating dynamic problems (motivating the present work), that spurt phenomena may be explained by differential constitutive laws that lead to a non-monotone relation of the total steady shear stress versus the steady shear strain-rate. In this framework, the increase of the volumetric flow rate corresponds to jumps in the strain rate when the driving pressure gradient exceeds a critical value. We conjecture that our stability result discussed in Sec. 3 below can be extended to these more complex problems.

2. Preliminaries

In this section, we discuss preliminary results that are essential for presenting the stability result; further details and proofs can be found in [8].

A. Well-Posedness.

We use abstract techniques of Henry [4] to study *global existence of classical solutions for smooth initial data of arbitrary size, and also existence of almost classical, strong solutions with discontinuities in the initial velocity gradient and in the stress components*. The latter result allows one to prescribe discontinuous initial data of the same type as the discontinuous steady states studied in this paper. Existence results of this type are established in [8] for a general class of problems that serve as models for shearing flows of non-Newtonian fluids; the total stress is decomposed into a Newtonian contribution and a finite number of stress relaxation components, viewed as internal variables that evolve in accordance with differential constitutive laws frequently used by rheologists (for discussion, formulation and results, see [11], [7], also the Appendix in [8]). Existence of classical solutions may also be obtained by using an approach based on the Leray - Schauder fixed point theorem (cf. Tzavaras [12] for existence results for a related system). Other existence results were obtained by Guillopé and Saut [2], and for models in more than one space dimension in [3].

As a consequence of the general theory, one obtains two global existence results (see Theorems 3.1, 3.2, 3.5, and Corollary 3.4 in [8]):

(i.) the existence of a unique classical solution $(v(x,t), \sigma(x,t))$ of (1.1)– (1.5) on $[0,1] \times [0,\infty)$ for initial data $(v_0(x), \sigma_0(x))$, not restricted in size, that satisfy: $S_0(x) :=$

$v_{0x}(x)+\sigma_0(x)+fx \in H^s[0,1]$ for some $s > 3/2$, with $S_0(0) = 0$, $v_0(1) = S_{0x}(1) = 0$, and $\sigma_0 \in C^1[0,1]$, where H^s denotes the usual interpolation space.

(ii.) the existence and uniqueness of a strong, "semi-classical" solution of (1.1) -(1.5), obtained by a different choice of function spaces, for initial data $(v_0(x), \sigma_0(x))$ that satisfy: $S_0(x) \in H^1[0,1]$ with $S_0(0) = 0$, $v_0(1) = S_{0x}(1) = 0$, and $\sigma_0 \in L^\infty[0,1]$.

Result (ii.) yields solutions in which σ and v_x may be discontinuous in x, but S_x and v_t are continuous, and σ is C^1 as a function of t for every x. Thus all derivatives appearing in the system may be interpreted *in a classical sense* as long as the equation is kept in conservation form. A result of this type was obtained by Pego in [10] for a different problem by a similar argument.

B. A Priori Bounds and Invariant Sets.

To discuss global boundedness of solutions, let σ, v be a classical solution on an arbitrary time-interval, and note that system (1.1) -(1.5) is endowed with the differential energy identity

$$\frac{d}{dt}\{1/2\int_0^1 v_t^2 dx + \int_0^1 [W(v_x) + x\, f\, v_x]dx\} + \int_0^1 [v_t^2 + v_{xt}^2]dx = 0\ . \tag{2.1}$$

The function $W(\xi) := \int_0^\xi w(\zeta)d\zeta$ plays the role of a stored energy function; by the assumption on g, W is not convex. *This fact is the main obstacle in the analysis of* stability.

(i.)*Boundedness of S.* Since $\xi g(\xi) > 0$, it follows that $\int_0^\xi g(\zeta)d\zeta \geq 0$ for $\xi \in \mathbb{R}$, and $W(\xi)$ satisfies the lower bound

$$W(\xi) + fx\xi \geq 1/4\xi^2 - f^2\,, \qquad \xi \in \mathbb{R}, \quad 0 \leq x \leq 1\,. \tag{2.2}$$

Standard energy estimates based on (2.1) and (2.2) coupled with integration of (1.1) with respect to x yield a global a priori bound for S:

$$|S(x,t)| \leq C \qquad 0 \leq x \leq 1, \quad 0 \leq t < \infty\,, \tag{2.3}$$

where C is a constant depending only on data but not t.

(ii.)*Invariant Sets for a Related ODE.* Control of S enables us to take advantage of the special structure of Eq. (1.2) and determine suitable invariant regions. For this purpose, it is convenient to introduce the quantity $s := \sigma + f\,x$. Then, Eqs. (1.2), (1.3) readily imply that s satisfies

$$s_t + s + g(s - S) = f\,x. \tag{2.4}$$

For a fixed x, it is convenient to view Eq. (2.4) as an ODE with forcing term $S(x, \bullet)$. Also, observe that at a steady state $(\overline{\sigma}, \overline{v}_x)$, one has $\overline{S} = 0$, and consequently, $\overline{s} = -\overline{v}_x$ is an equilibrium solution of (2.4) (with $S = 0$). If $S \equiv 0$ in (2.4), the hypothesis concerning g implies that the ODE admits positively invariant intervals for each fixed x. We sketch how this property is preserved in the presence of *a priori control of S* as provided by (2.3); more delicate bounds are essential in the proof of stability in Sec .3.

To fix ideas, let $t_0 > 0$ be given, and assume that

$$|S(x,t)| \le \rho\,, \quad 0 \le x \le 1\,, 0 \le t \le t_0\,, \tag{2.5}$$

for some $\rho > 0$. For x fixed in $[0,1]$, we use the notation $S(t) := S(x,t)$ and conveniently rewrite (2.4) as

$$s_t + w(s - S(t)) = f\,x - S(t)\,. \tag{2.6}$$

We state the following result on invariant intervals; its proof is obvious.

Proposition 2.1. *Let S satisfy the uniform bound* (2.5) *for* $0 \le t \le t_0$. *For x fixed,* $0 \le x \le 1$, *assume there exist s_-, s_+ such that $s_- < s_+$ and*

$$w(s_- - \lambda) < f\,x - \lambda \quad , \quad |\lambda| \le \rho \tag{2.7}$$

$$w(s_+ - \lambda) > f\,x - \lambda \quad , \quad |\lambda| \le \rho \tag{2.8}$$

Then the compact interval $[s_-, s_+]$ *is positively invariant for the ODE* (2.6) *on the time interval* $0 \le t \le t_0$.

Invariant intervals are generated by solution sets of the inequalities (2.7) and (2.8) as functions of ρ and x. In particular, since $\lim_{\xi \to \pm\infty} w(\xi) = \pm\infty$, given any x and ρ, one easily determines s_{0+} large, positive and s_{0-} large, negative such that if $s_- < s_{0-}$ and $s_+ > s_{0+}$, then s_- and s_+ satisfy (2.7) and (2.8), respectively, and the compact interval $[s_-, s_+]$ is positively invariant for the ODE (2.6).

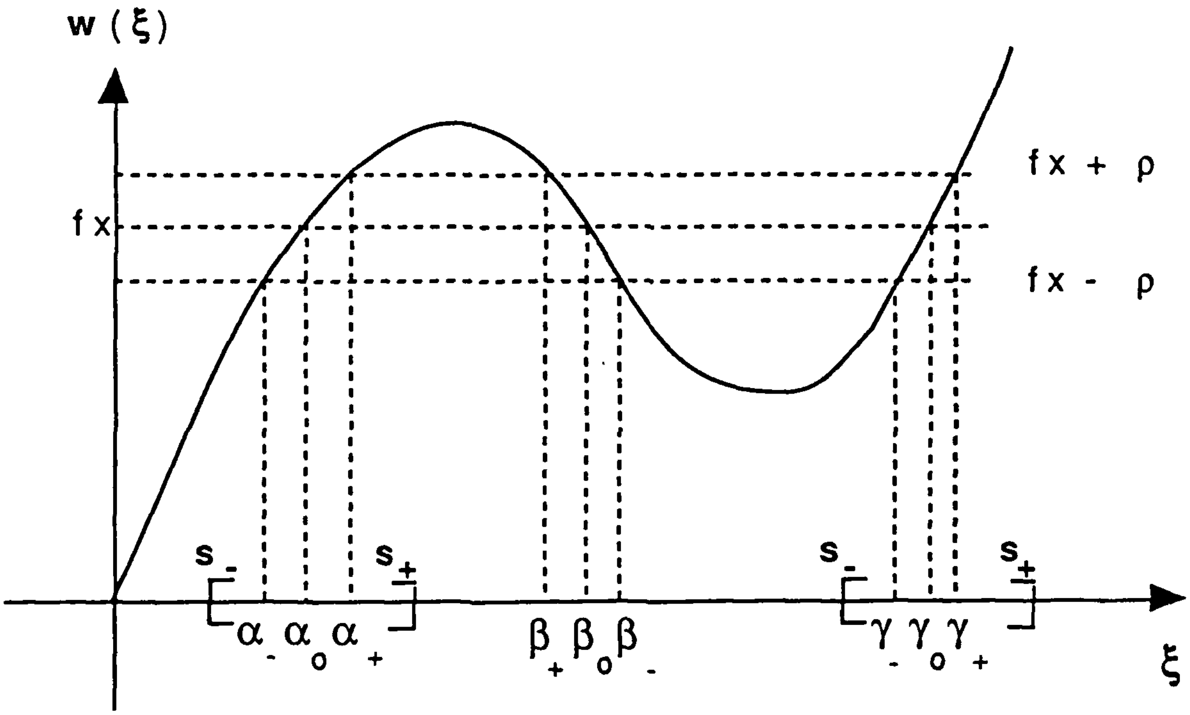

Fig. 3: Invariant Intervals.

More discriminating choices of invariant intervals occur if one restricts attention to small values of ρ; the analysis becomes more delicate. For a function $w(\xi)$ with a single loop, the most interesting case arises when $f\,x-\rho$, $f\,x$ and $f\,x+\rho$ each intersects the graph of $w(\xi)$ at three distinct points. Referring to Fig. 3, the abscissae of the points of intersection are denoted by $(\alpha_-, \beta_-, \gamma_-)$, $(\alpha_0, \beta_0, \gamma_0)$ and $(\alpha_+, \beta_+, \gamma_+)$, respectively. It turns out that for x fixed and ρ small enough, there are discriminating invariant intervals of the type shown in Fig. 3. However, in contrast to the large invariant intervals discussed in the previous paragraph, the more discriminating ones degenerate as we approach the top or bottom of the loop (when x varies).

For the stability of discontinuous steady states in Sec. 3., it is crucial to construct compact invariant intervals that are of uniform length (see Corollary 2.2 in [8]). The latter is accomplished by taking ρ sufficiently small and by avoiding the top and bottom of the loop in Fig. 3. Of specific interest is the situation in which $\overline{s}(x)$ is a piecewise smooth solution of

$$w(\overline{s}(x)) = f\,x\,, \tag{2.9}$$

defined on $[0,1]$ and admitting jump discontinuities at a finite number of points $x_1, ..., x_n$ in $[0,1]$. Recall that $\overline{s}(x)$ is a steady solution of the ODE (2.6) corresponding to the steady state $(\overline{\sigma}, \overline{v}_x)$. In addition, suppose that $\overline{s}(x)$ takes values in the monotone increasing parts of the curve $w(\xi)$ and that it avoids jumping at the top or bottom of the loop, i.e.,

$$w'(\overline{s}(x)) \geq c_0 > 0\,, \qquad x \in [0,1] \setminus \{x_1, ..., x_n\}\,, \tag{2.10}$$

for some constant c_0. A delicate construction in [8] yields compact, positively invariant intervals of (2.6) of uniform length, centered around $\overline{s}(x)$ at each $x \in [0,1] \setminus \{x_1, ..., x_n\}$.

(iii.)*Boundedness of σ and v_x.* As an easy application of Sec. 2 *(ii)*, choose a compact interval $[s_-, s_+]$ that is positively invariant for (2.6) and valid for all $x \in [0,1]$. By virtue of the global bound (2.3) satisfied by $S(x,t)$, we conclude that

$$|s(x,t)| \leq C\,, \qquad 0 \leq x \leq 1,\, t \geq 0 \tag{2.11}$$

which, in turn, using (1.3) and (2.11), implies

$$|v_x(x,t)| \leq C\,, \qquad 0 \leq x \leq 1,\, t \geq 0\,, \tag{2.12}$$

for some constant C depending only on the data. The definition of s also implies that σ is uniformly bounded.

(iv.)*Convergence to steady states.* Let $(v(x,t), \sigma(x,t))$ be a classical solution of (1.1) -(1.5) defined on $[0,1] \times [0,\infty]$. We discuss the behavior of this solution as $t \to \infty$.

The first result indicates that $S = \sigma + v_x + f\,x$ converges to its equilibrium value.

Proposition 2.2. *Under the assumptions of the existence results,*

$$\lim_{t \to \infty} S(x,t) = 0\,, \tag{2.13}$$

uniformly for $x \in [0,1]$.

The proof is a consequence of Sobolev embedding applied to the following a priori estimates that are derived from the system (1.1)–(1.4) by standard techniques:

$$\int_0^\infty \int_0^1 S_t^2 dx d\tau \le C \,, \tag{2.14}$$

$$\int_0^\infty \int_0^1 S^2 dx d\tau \le C \,, \tag{2.15}$$

$$\int_0^1 S_x^2(x,t) dx \le C \,,\; 0 \le t < \infty \,, \tag{2.16}$$

where C is a positive constant depending only on data.

Use of (2.13) enables us to identify the limiting behavior of solutions of (2.4) as $t \to \infty$. The following result is analogous to Lemma 5.5 in Pego [10]; its elementary proof is given in Lemma 4.2 of [8].

Proposition 2.3. *Let $s(x,\bullet) \in C^1[0,\infty)$ be the solution of* (2.10), *where $S(x,\bullet)$ is continuous and satisfies* (2.13), $0 \le x \le 1$. *Then $s(x,\bullet)$ converges to $s^\infty(x)$* as $t \to \infty$ *and $s^\infty(x)$ satisfies*

$$s^\infty(x) + g(s^\infty(x)) = f\,x \quad , \quad 0 \le x \le 1 \,. \tag{2.17}$$

In view of the shape of $w(\xi) = \xi + g(\xi)$, equation (2.17) has one solution for $0 \le f\,x < m$ or $f\,x > M$ and three solutions for $m < f\,x < M$.

Let $(v(x,t), \sigma(x,t))$ be a classical solution of (1.1)–(1.4) on $[0,1] \times [0,\infty)$. Recalling the definition of s, Proposition 2.3 implies

$$\sigma^\infty(x) = \lim_{t\to\infty} \sigma(x,t) = s^\infty(x) - f\,x \,. \tag{2.18}$$

Also, combining (1.1), (2.13) and (2.18) yields

$$v_x^\infty(x) := \lim_{t\to\infty} v_x(x,t) = \lim_{t\to\infty} (S(x,t) - s(x,t)) = -s^\infty(x) \,, \tag{2.19}$$

and

$$S^\infty(x) = v_x^\infty(x) + \sigma^\infty(x) + f\,x = 0 \,. \tag{2.20}$$

Finally, noting that

$$v(x,t) = -\int_x^1 v_x(x,t) dx \,, \tag{2.21}$$

$v^\infty(x)$ is Lipschitz continuous and satisfies

$$v^\infty(x) := \lim_{t\to\infty} v(x,t) = \int_x^1 s^\infty(\xi) d\xi \,. \tag{2.22}$$

We conclude that any solution of (1.1)–(1.4) converges to one of the steady states. If $0 \leq f < m$, then there is a unique smooth steady state which is the asymptotic limit of any solution. However, if $m < f$, then there are multiple steady states and thus a multitude of possible asymptotic limits. In Sec. 3, we identify stable steady states. Also note from (2.20) that in a discontinuous steady state, the discontinuities in $\overline{\sigma}$ and $\overline{v}_x$ cancel.

Observe that in case $w(\xi)$ is monotone the above arguments yield that every solution converges to the unique steady state. Moreover, the above results can be routinely generalized to the case that the function $w(\xi)$ has multiple loops but the graph of w has no horizontal segments.

3. Stability of Steady States

The purpose is to study the stability of velocity profiles with kinks. To fix ideas, let $(\overline{v}(x), \overline{\sigma}(x))$ be a steady state of (1.1)–(1.3) such that $\overline{v}(x)$ has a finite number of kinks located at the points $x_1, \ldots, x_n$ in $(0,1)$; accordingly, $\overline{v}_x(x)$ and $\overline{\sigma}(x)$ have a finite number of jump discontinuities at the same points. Recall that, if we set $\overline{u}(x) = -\overline{v}_x(x)$,

$$w(\overline{u}(x)) = fx\,, \quad x \in [0,1],\, x \neq x_1, \ldots, x_n \tag{3.1}$$

and $\overline{\sigma}(x) = g(-\overline{u}(x))$.

Given smooth initial data $(v_0(x), \sigma_0(x))$, there is a unique smooth solution $(v(x,t), \sigma(x,t))$ of (1.1)–(1.4). As $t \to \infty$, the solution converges to one of the steady states, not a-priori identifiable. We now restrict attention to initial data that are close to $(\overline{v}(x), \overline{\sigma}(x))$, except on the union $\mathcal{U}$ of small subintervals centered around the points $x_1, \ldots, x_n$. $\mathcal{U}$ can be thought of as the location of transition layers separating the smooth branches of the steady state. Roughly speaking, it turns out that the steady state is "asymptotically stable" under smooth perturbations that are close in energy, provided $(\overline{v}(x), \overline{\sigma}(x))$ takes values in the monotone increasing parts of $w(\xi)$; the stable solutions are local minimizers of an associated energy functional (see (3.8) below). The interesting problem of finding the domain of attraction of a stable steady solution appears to be a difficult task. Our main result is:

Theorem 3.1. *Let $(\overline{v}(x), \overline{\sigma}(x))$ be a steady state solution as described above and satisfying*

$$w'(\overline{v}_x(x)) \geq c_0 > 0\,,\ x \in [0,1], \qquad x \neq x_1, \ldots, x_n \tag{3.2}$$

for some positive constant c_0. If the measure of $\mathcal{U}$ is sufficiently small, there is a positive constant δ_0 depending on $\mathcal{U}$ such that, if $\delta < \delta_0$, then for any initial data $(v_0(x), \sigma_0(x))$ satisfying

$$\sup_{0 \leq x \leq 1} |S_0(x)| < \delta, \tag{3.3}$$

$$\int_0^1 v_t^2(x,0)dx < \frac{1}{2}\delta^2 \tag{3.4}$$

and

$$|v_{0x}(x) - \overline{v}_x(x)| < \delta \quad , \quad x \in [0,1] \setminus \mathcal{U} \tag{3.5}$$

the corresponding solution $(v(x,t), \sigma(x,t))$ *approaches the steady state* $(\overline{v}(x), \overline{\sigma}(x))$ *as* $t \to \infty$, *in the sense,*

$$v_x(x,t) \to \overline{v}_x(x)\,, \tag{3.6}$$

$$\sigma(x,t) \to \overline{\sigma}(x)\,, \tag{3.7}$$

for all $x \in [0,1] \setminus \mathcal{U}$.

The above result is similar, in spirit and approach, to the analysis of Andrews and Ball [1], and particularly to stability results established by Pego [10] for motions of one-dimensional viscoelastic materials of rate type with a non-monotonic stress-strain relation. Current work of Novick-Cohen and Pego [9] on spinodal decomposition involve a similar stability analysis.

Because the argument is lengthy and delicate, we can only indicate the main idea of the proof for the case that $\overline{u}(x) = -\overline{v}_x(x)$ has one single jump discontinuity located at x_0, $m < fx_0 < M$, and for $\mathcal{U} = (x_0 - \varepsilon, x_0 + \varepsilon)$ for some small ε. Minor modifications are needed to account for the general case. Technical details can be found in [8], Theorem 5.1. The proof is based on exploiting the energy identity (2.1), which, upon setting $u(x,t) = -v_x(x,t)$ and integrating with respect to t, yields the inequality

$$\begin{aligned} &\frac{1}{2}\int_0^1 v_t^2(x,t)dx + \int_0^1 [(W(u(x,t)) - xfu(x,t)) - \Phi(x)]dx \\ &\qquad \le \frac{1}{2}\int_0^1 v_t^2(x,0)dx + \int_0^1 [(W(u_0(x)) - xfu_0(x) - \Phi(x)]dx; \end{aligned} \tag{3.8}$$

note that the integral of the function Φ has been subtracted from both sides of (3.8). The function $\Phi(x)$ is associated with the particular choice of the discontinuous steady state $\overline{u}(x)$, the stability of which is being tested. The function Φ identifies a basin of attraction of the state $\overline{u}(x)$. Roughly speaking, the goal is to find Φ so that the second integral on the left side of (3.8) is positive, and at the same time the right side can be made sufficiently small. The construction of Φ is delicate because W in (3.8) is not convex, and because the double-well potential $W(u) - xfu$ depends explicitly on x. Note that for each fixed x, the function $\overline{u}(x)$ is the horizontal coordinate of the bottom of one of the wells; the left well if $x < x_0$, the right well if $x > x_0$. To insure that the construction of Φ produces the property desired, this part of the analysis makes crucial use of invariant intervals of the ODE (2.6) that are of uniform length as discussed in Sec. 2(ii) above.

References

1. G. Andrews and J. Ball, "Asymptotic Stability and Changes of Phase in One-Dimensional Nonlinear Viscoelasticity:," *J. Diff. Eqns.* **44** (1982), pp. 306–341.

2. C. Guillopé and J.-C. Saut, "Global Existence and One-Dimensional Nonlinear Stability of Shearing Motions of Viscoelastic Fluids of Oldroyd Type," *Math. Mod. Numer. Anal.*, 1990. To appear.

3. C. Guillopé and J.-C. Saut, "Existence Results for Flow of Viscoelastic Fluids with a Differential Constitutive Law," *Math. Mod. Numer. Anal.*, 1990. To appear.

4. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, vol. 840 Springer-Verlag, New York, 1981.

5. J. Hunter and M. Slemrod, "Viscoelastic Fluid Flow Exhibiting Hysteretic Phase Changes," *Phys. Fluids* **26** (1983), pp. 2345–2351.

6. D. Malkus, J. Nohel, and B. Plohr, "Dynamics of Shear Flow of a Non-Newtonian Fluid," *J. Comput. Phys.*, 1989. To appear.

7. D. Malkus, J. Nohel, and B. Plohr, "Analysis of New Phenomena In Shear Flow of Non-Newtonian Fluids," in preparation, 1989.

8. J. Nohel, R. Pego, and A. Tzavaras, "Stability of Discontinuous Steady States in Shearing Motions of Non-Newtonian Fluids," *Proc. Roy. Soc. Edinburgh, Series A*, 1989. submitted.

9. A. Novick-Cohen and R. Pego, "Stable Patterns in a Viscous Diffusion Equation," *preprint*, 1989. submitted.

10. R. Pego, "Phase Transitions in One-Dimensional Nonlinear Viscoelasticity: Admissibility and Stability," *Arch. Rational Mech. and Anal.* **97** (1987), pp. 353–394.

11. M. Renardy, W. Hrusa, and J. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 35, Longman Scientific & Technical, Essex, England, 1987.

12. A. Tzavaras, "Effect of Thermal Softening in Shearing of Strain-Rate Dependent Materials," *Arch. Rational Mech. and Anal.* **99** (1987), pp. 349–374.

13. G. Vinogradov, A. Malkin, Yu. Yanovskii, E. Borisenkova, B. Yarlykov, and G. Berezhnaya, "Viscoelastic Properties and Flow of Narrow Distribution Polybutadienes and Polyisoprenes," *J. Polymer Sci., Part A-2* **10** (1972), pp. 1061–1084.